The positivity and other properties of the matrix of capacitance: physical and mathematical implications

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Abstract

We prove that the matrix of capacitance in electrostatics is a positive-singular matrix with a non-degenerate null eigenvalue. We explore the physical implications of this fact, and study the physical meaning of the eigenvalue problem for such a matrix. Many properties are easily visualized by constructing a "potential space" isomorphic to the euclidean space. The problem of minimizing the internal energy of a system of conductors under constraints is considered, and an equivalent capacitance for an arbitrary number of conductors is obtained. Moreover, some properties of systems of conductors in successive embedding are examined. Finally, we discuss some issues concerning the gauge invariance of the formulation.

Keywords: Capacitance, electrostatics, positive matrices, eigenvalue problems, boundary conditions.

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1 Introduction

The concept of capacitance and the matrix of capacitance has been studied from several points of view [1]-[15]. On the other hand, the theory of positive matrices and operators is extensively used in branches of Physics such as the mechanics of rigid body motion, quantum mechanics [21, 22], and other more advanced topics [16]-[20]. Nevertheless, the employment of the theory of matrices and operators to study the matrix of capacitance is rather poor [23]-[26]. In particular, no physical meaning is usually given to the eigenvalue problem of the matrix of capacitance. We shall see that the theory of positive matrices and operators could provide another point of view that enlighten many mathematical and physical properties of systems of electrostatic conductors. The paper is distributed as follows: section 2 defines the electrostatic system of conductors that we intend to study, and establish the notation and properties necessary for our subsequent developments. In Sec. 3 along with Appendix A, the main goal is to prove the positivity of the matrix of capacitance. Sec. 4 discusses some subtleties with respect to the gauge invariance of the formulation. Section 5 along with appendix B explores the physical implications of the positivity of the matrix of capacitance. This is done by constructing a "space of potentials" with inner product in which the matrix of capacitance represents an hermitian positive operator. Section 6 studies the problem of minimization of the internal energy of a system with constraints, and an equivalent capacitance is defined for a system with arbitrary number of conductors. On the other hand, configurations of conductors that are successively embedded deserves special attention because many simplifications are posible, and this is the topic of Sec. 7 and appendix C. Section 8 summarizes our conclusions and appendix D contains suggested problems for readers.

2 Basic Framework

This section summarizes some properties of the matrix of capacitance obtained in Ref. [27]. They are the framework of our developments in the remaining sections. Let us consider a system of N conductors and an equipotential surface that surrounds them, such equipotential surface could be the cavity of an external conductor. The potential on each internal conductor is denoted by φ_i, $i = 1, 2, \ldots, N$. (see Fig. 1). We define a set of surfaces S_i sligthly bigger than the surfaces of the conductors and locally parallel to them, $\mathbf{n}_i$ is an unit vector normal to the surface S_i pointing outward with respect to the conductor. The potential of the equipotential surface is denoted by φ_{N+1} and we define a surface S_{N+1} slightly smaller and locally parallel to the surface of the equipotential. The charges on the conductors are denoted by Q_i with $i = 1, ..., N$ and if there is a cavity of an external conductor in the equipotential surface we denote the charge accumulated in such a cavity by Q_{N+1}, the unit vector $\mathbf{n}_{N+1}$ points inward with respect to the equipotential surface. Finally, we define the total surface $S_T = S_1 + \ldots + S_{N+1}$ and the volume V_{S_T} defined by the surface S_T i.e. the volume delimited by the external surface S_{N+1} and the N internal surfaces S_i.

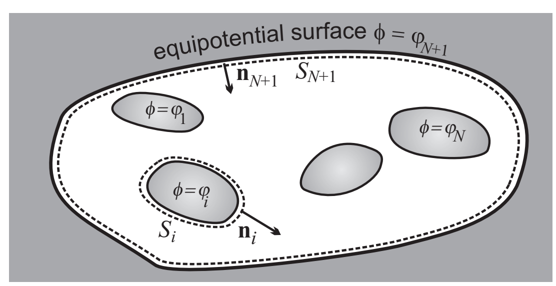

Figure 1: N conductors surrounded by an equipotential surface. The volume V_{S_T} is the region in white.

Let us define a set of dimensionless auxiliary functions f_i that obey Laplace's equation in the volumen V_{S_T} with the boundary conditions

$$\nabla^2 f_j = 0, \quad f_j(S_i) = \delta_{ij}, \;\; (i, j = 1, \ldots, N+1). \tag{1}$$

The uniqueness theorem ensures that the solution for each f_j is unique in V_{S_T}. The boundary conditions (1) indicate that the f_j functions depend only on the geometry. Since the functions f_j acquire constant values on the surfaces S_i with $i =$

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$1, \ldots, N+1$, it is clear that ∇f_j is orthogonal to these surfaces. The functions f_j have some properties [27]

$$\sum_{j=1}^{N+1} f_j = 1;\ \nabla f_j\left(S_i\right) \cdot \mathbf{n}_i = \left(1-2\delta_{ij}\right) \left\|\nabla f_j\left(S_i\right)\right\|;\ 0 \leq f_j \leq 1 \tag{2}$$

From these auxiliary functions we can construct a matrix that provides a linear relation between the set of charges Q_i and the set of potentials φ_i in the following way

$$C_{ij} \equiv -\varepsilon_0 \oint_{S_i} \nabla f_j \cdot \mathbf{n}_i\, dS = \varepsilon_0 \int_{V_{S_T}} \nabla f_i \cdot \nabla f_j\, dV \tag{3}$$

$$Q_i = \sum_{j=1}^{N+1} C_{ij}\varphi_j \tag{4}$$

and some properties of the C_{ij} matrix can be derived

$$C_{ij} = C_{ji},\quad \sum_{j=1}^{N+1} C_{ij} = \sum_{i=1}^{N+1} C_{ij} = 0, \tag{5}$$

$$C_{ii} \geq 0\ ,\ C_{ij} \leq 0, \qquad (i \neq j). \tag{6}$$

The equations above are valid for $i,j = 1,\ldots,N+1$. The expressions below are valid for $i,j = 1,\ldots,N$

$$\sum_{i=1}^{N} C_{i,N+1} \leq 0\ ,\ \sum_{i=1}^{N} C_{ij} \geq 0 \tag{7a}$$

$$|C_{jj}| \geq \sum_{i \neq j}^{N} |C_{ij}|,\quad C_{ii}C_{jj} \geq C_{ij}^2 \tag{7b}$$

$$|C_{N+1,N+1}| = \sum_{i=1}^{N} |C_{i,N+1}|, \tag{7c}$$

$$|C_{N+1,N+1}| \geq |C_{i,N+1}| \tag{7d}$$

and expressions for the internal electrostatic energy U of the system and of the reciprocity theorem can be obtained

$$U = \frac{1}{2}\sum_{i,j}^{N+1} C_{ij}\varphi_j\varphi_i = \frac{1}{2}\sum_{i}^{N+1} Q_i\varphi_i;\ \sum_{i=1}^{N+1} Q_i\varphi_i' = \sum_{j=1}^{N+1} Q_j'\varphi_j \tag{8}$$

where $\{Q_i, \varphi_i\}$ and $\{Q_i', \varphi_i'\}$ are two sets of charges and potentials over the same configuration of conductors. The C_{ij} elements constitute a real symmetric matrix of dimension $(N+1) \times (N+1)$, in which the number of degrees of freedom is $N(N+1)/2$, note that it is the same number of degrees of freedom of a $N \times N$ real symmetric matrix.

For future purposes, we shall call the matrix with elements C_{ij} and with $i,j = 1,\ldots,N$ the r-matrix (restricted matrix denoted by $\mathbf{C}$), while the C_{ij} matrix with $i,j = 1,\ldots,N+1$ will be called the e-matrix (extended matrix denoted by $\mathbf{C}_e$).

3 Discussion of the mathematical properties of the matrix

Equations (1) and (3) tell us that the C_{ij} elements are purely geometrical. In addition, Eqs. (3) and (5) say that the e-matrix is a real symmetric matrix in which the sum of elements of each row and column is null. From Eq. (6) the non-diagonal elements of the e-matrix are non-positive. The volume integral in Eq. (3) shows that the diagonal elements C_{kk} are strictly positive for any well-behaved geometry. In particular, since $C_{N+1,N+1}$ is positive, Eq. (7c) shows that at least one element of the form $C_{i,N+1}$ is different from zero (negative) for $i = 1,\ldots,N$; thus rewriting Eq. (5) in the form

$$\sum_{j=1}^{N} C_{ij} = -C_{i,N+1} \tag{9}$$

we see that if $C_{i,N+1} < 0$ the sum of the elements of the $i-$row of the r-matrix is positive, if $C_{i,N+1} = 0$ such a sum is null. Since at least one of the $C_{i,N+1}$ elements is strictly negative, we conclude that in the r-matrix the sum of elements on each row is non-negative and for at least one row the sum is positive. Because of the symmetry, all statements about rows are valid for columns.

On the other hand, when V_{S_T} is a connected region as in Fig. 1, the function f_j should change progressively from its value 1 on conductor j up to the value zero in the conductor i without taking local minima or maxima according to the properties of Laplace's equation. According with Eq. (2) the factor $\nabla f_j\left(S_i\right) \cdot \mathbf{n}_i$ is positive for $i \neq j$ and from Eq. (3) the non-diagonal C_{ij} factors must be strictly negative for a well-behaved geometry. This discussion is not valid when the volume V_{S_T} is non-connected as in Fig. 2, we shall discuss this case in section 7. When $C_{ij} < 0$ for $i \neq j$, the discussion below Eq. (9), leads to the fact that the sum of elements in each row of the r-matrix is positive.

In conclusion, for the e-matrix the sum of elements of each row is null. Further, if V_{S_T} is a connected region, all matrix elements of the e-matrix are non-null (for a well-behaved geometry), and for the r-matrix the sum of elements of each row is positive. Theorems **A** and **B** in Appendix A, show that under these conditions we find: ❶ The e-matrix is a real singular positive matrix, its null eigenvalue is non-degenerate and the other eigenvalues are positive. ❷ The r-matrix is a real positive-definite matrix*. Its eigenvalues are all positive. ❸ The null eigenvalue of the e-matrix is associated with $(N+1)-$dimensional eigenvectors of the form

$$\phi_0^T \equiv (\varphi_0, \varphi_0, ..., \varphi_0) \tag{10}$$

4 Gauge invariance of the formulation

We have two possible scenarios here, in the first the equipotential surface is the surface of the cavity of a conductor that encloses the others. In the second, the equipotential surface is just a geometrical place in the vacuum. The uniqueness theorem guarantees the same solution in both cases but only in the interior of the equipotential surface. In the equipotential surface itself we can see that in the first case there is a charge Q_{N+1} accumulated in the cavity, while in the second case there is no charge in such a surface at all. The problem lies in the fact that the electric field is not well-behaved in the surface of the cavity because of the accumulation of surface charge [23]-[26], it is precisely because of this fact that we defined surfaces slightly different from the real surfaces on each conductor (in which ∇f_j are well-defined). So all the observables (charges, potentials, electric fields) are the same in the interior of the equipotential surface for both scenarios, but the surface charge and the electric field differ in both

*The non-degeneration of the null eigenvalue of the e-matrix follows from theorem **A** or alternatively from theorem **B**, in appendix A, after establishing the positive -definite nature of the r-matrix.

cases when they are evaluated on the equipotential surface itself[†]. Anyway, the internal charges and any other observables not defined on the equipotential surface, are calculated in both scenarios with the same set of C_{ij} coefficients.

From the discussion above, we see that when we have a set of free conductors, the simplest equipotential surface that we can define is the one lying at infinity with zero potential, which is equivalent for most of the purposes to consider a cavity of a grounded external conductor in which all the dimensions of the cavity tend to infinity.

Further, we shall see that the linear relation between charges and potentials in Eq. (4) is gauge invariant by shifting the potential throughout the space as $\varphi' \to \varphi + \varphi_0$ with φ_0 being a non-zero constant. This gauge transformation must keep all observables unaltered, in particular the charge Q_k on each surface of the conductors. Writing Eq. (4) in matrix form and using Eq. (10) we have

$$\mathbf{Q}' = \mathbf{C}_e\left(\phi + \phi_0\right) = \mathbf{C}_e\phi = \mathbf{Q} \tag{11}$$

where we used the fact that ϕ_0 is an eigenvector of $\mathbf{C}_e$ with null eigenvalue. This gauge invariance says that there is an infinite number of solutions (sets of potentials) for the linear equations (4) with given values of the charges, this fact is related in turn with the non-invertibility of $\mathbf{C}_e$. In other words, gauge invariance is related with the existence of an eigenvector with null eigenvalue which is also equivalent to the non-invertibility. On the other hand, the singularity of a matrix is also related with the linear dependence of the column (or row) vectors that constitute the matrix, this lack of independence in the case of $\mathbf{C}_e$ is manifested in the fact that no all charges can be varied independently as can be seen from the expression

$$Q_{int} = -Q_{N+1} \tag{12}$$

where Q_{int} is the total charge of the internal conductors while Q_{N+1} is the charge accumulated on the surface of the cavity of the external conductor[‡]. Further, the linear dependence of the e-matrix can be visualized by observing that it has the same degrees of freedom as the r-matrix. This fact induces us to find expressions involving the r-matrix only. For this, we can rewrite Eq. (4) by following the procedure that leads to Eq. (39)

$$Q_k = \sum_{m=1}^{N} C_{km}\left(\varphi_m - \varphi_{N+1}\right) \equiv \sum_{m=1}^{N} C_{km}V_m \tag{13}$$

these relations are valid for $k = 1, \ldots, N+1$. However, since Eq. (12) shows that Q_{N+1} is not independent, we can restrict them to $k = 1, \ldots, N$. Rewriting Eq. (13) in matrix form with this restriction we get

$$\mathbf{Q} = \mathbf{CV}; \quad \mathbf{V} \equiv \left(V_1, V_2, ..., V_N\right), \quad V_i \equiv \varphi_i - \varphi_{N+1} \tag{14}$$

this relation is written in terms of voltages instead of potentials, so it is clearly gauge invariant. Further, the relation is invertible because the r-matrix $\mathbf{C}$ is positive-definite. It worths emphasizing that all expressions obtained from now on in terms of voltages and the r-matrix, are valid only if the voltages are taken with respect to the φ_{N+1} potential.

[†]Of course the potential on the equipotential surface is the same in both cases by definition.

[‡]This can be shown from Gauss's law or directly from the formalism presented here (see Ref. [27]). If the equipotential surface is a geometrical place in the vacuum, Eq. (12) must be interpreted as a numerical equality between the total internal charge and the quantity on the right-hand side of Eq. (4) with $i = N+1$.

5 Physical implications of the positivity of the matrix

To facilitate the derivation and interpretation of the results let us define the following quantities

$$c_{ij} \equiv \frac{1}{k_0}C_{ij} \quad ; \quad \Phi_i \equiv \frac{1}{k_0}Q_i$$

where k_0 is a constant defined such that c_{ij} are dimensionless. From these definitions Eq. (4) could be rewritten in the form

$$\Phi_i = \sum_{i=1}^{N+1} c_{ij}\varphi_j \quad ; \quad \mathbf{\Phi} = \mathbf{c}_e\phi \tag{15}$$

the dimensionless c_{ij} factors contain the same information as C_{ij}. Similarly, Φ_i are quantities with dimension of potential but with the physical information of the charges Q_i (it is like a "natural unit" for the charge). The aim of settle the charges and potentials with the same dimension is to interpret Eq. (15) as a linear transformation in the configuration space Φ^{N+1} in which each axis has dimensions of potential. This space would be isomorphic to R^{N+1} if we define an inner product of the form

$$\left(\mathbf{\Phi}, \phi\right) = \mathbf{\Phi}^\dagger\phi = \sum_{i=1}^{N+1}\Phi_i\phi_i$$

where we have taken into account that this is a real vector space. The capacitance matrix is hermitian (real and symmetric) with respect to this inner product. Now let us take two sets of charges and potentials $\{\mathbf{\Phi}, \phi\}$ and $\{\mathbf{\Phi}', \phi'\}$ over the same configuration of conductors. Doing the inner product $(\mathbf{\Phi}', \phi)$, using Eq. (15) and taking into account the hermiticity of $\mathbf{c}_e$, we have

$$\left(\mathbf{\Phi}', \phi\right) = \left(\mathbf{c}_e\phi', \phi\right) = \left(\phi', \mathbf{c}_e\phi\right) = \left(\phi', \mathbf{\Phi}\right) = \left(\mathbf{\Phi}, \phi'\right)$$

so that

$$\left(\mathbf{\Phi}', \phi\right) = \left(\mathbf{\Phi}, \phi'\right)$$

which is the reciprocity theorem. From this point of view, this theorem is a manifestation of the hermiticity of the e-matrix. Of course, we can define a potential space $\mathbf{\Phi}^N$, in which the N internal charges and N voltages form $N-$dimensional vector arrangements and the r-matrix acts as an hermitian operator. In this space the reciprocity theorem acquires the form

$$\left(\mathbf{\Phi}', \mathbf{V}\right) = \left(\mathbf{\Phi}, \mathbf{V}'\right)$$

where in this case $\mathbf{\Phi}'$ and $\mathbf{\Phi}$ refer to configurations of the internal charges only. Now we shall rewrite the electrostatic internal energy U of the system given by Eq. (8) in our new language

$$u = \frac{1}{2}\left(\phi, \mathbf{c}_e\phi\right) \geq 0 \quad ; \quad u \equiv U/k_0 \tag{16}$$

the inequality comes from the positivity of the e-matrix. This expression is gauge invariant and can be written in terms of the r-matrix and voltages (see appendix B)[§] as follows

$$u = \frac{1}{2}\left(\mathbf{V}, \mathbf{cV}\right) \geq 0$$

[§]There is a subtlety with the concept of internal energy. The value of an energy is not gauge invariant, but the internal energy is indeed a difference of energies between an initial and a final configuration (or a work to ensemble a given system) this value should then be gauge invariant.

Because $\mathbf{c}$ is positive-definite, a zero energy is obtained only with $\mathbf{V} = \mathbf{0}$. The only configurations with zero energy are the ones with all potentials equal[¶]. Hence, for any geometry of the set of conductors and for any configuration of charges and potentials on them, the external agent that ensembles it, makes a net work on the system. There is no configuration in which the system makes a net work on the external agent. Note that all the analysis above is consistent with the features coming from the equivalent equation

$$u = \frac{1}{2}\int_{V_{S_T}} \mathbf{E}^2 \, dV$$

where $\mathbf{E}$ is the electric field generated by the configuration throughout the volume V_{S_T}.

Let us interpret the eigenvalue equation of $\mathbf{c}$. It reads

$$\mathbf{c}\mathbf{V}^{(k)} = \lambda_k \mathbf{V}^{(k)} \Rightarrow \Phi^{(k)} = \lambda_k \mathbf{V}^{(k)}$$

we use superscripts to label a given eigenvector and subscripts to label a given component of a fixed eigenvector, if there is a set $\{i\}$ of n indices such that all the λ_i's are identical, this eigenvalue is $n-$fold degenerate. Each eigenvector $\mathbf{V}^{(k)}$ means a configuration of voltages for which each internal charge $\Phi_i^{(k)}$ is related with its corresponding voltage $V_i^{(k)}$ by the same constant of proportionality λ_k. Now, since the eigenvalues are positive, each internal charge $\Phi_i^{(k)}$ and its corresponding voltage $V_i^{(k)}$ have the same sign.

Let us construct a complete orthonormal set of real dimensionless eigenvectors $\mathbf{u}^{(k)}$ of $\mathbf{c}$ associated with the eigenvalues λ_k. We show in appendix B, Eq. (43) that the internal energy associated with a set of voltages described by the vector $\mathbf{V}$ can be written in terms of those eigenvectors and eigenvalues

$$u = \frac{1}{2}\sum_{n=1}^{N} \lambda_n \left|\left(\mathbf{u}^{(n)}, \mathbf{V}\right)\right|^2$$

The set $\left\{\mathbf{u}^{(n)}\right\}$ defines principal axes in the potential space Φ^N, and $\left(\mathbf{u}^{(n)}, \mathbf{V}\right)$ is the projection of the vector $\mathbf{V}$ along with the principal axis $\mathbf{u}^{(n)}$. If the configuration of voltages in the system is of the form $\mathbf{V}^{(k)} = V_0 \mathbf{u}^{(k)}$ (i.e. if the vector $\mathbf{V}$ is parallel to a principal axis) we find[‖]

$$u = \frac{1}{2}\lambda_k V_0^2 = \frac{1}{2}\lambda_k \left\|\mathbf{V}^{(k)}\right\|^2$$

so the eigenvalue is proportional to the internal energy associated with a set of voltages that forms the corresponding normalized eigenvector of the r-matrix.

6 Minimization of the internal energy

Let us find the configuration $\mathbf{V}$ of voltages that minimizes the internal energy with the constraint that the total internal charge Q_{int} is a constant Q_0. Since $Q_{int} = -Q_{N+1}$ and taking into account that Eq. (13) is also valid for $k = N+1$, we have

$$Q_{int} = -\sum_{j=1}^{N} C_{N+1,j} V_j = Q_0 \tag{17}$$

the function $Z(\mathbf{V})$ that defines the constraint is

$$Z(\mathbf{V}) \equiv -\sum_{j=1}^{N} C_{N+1,j} V_j - Q_0 = 0 \tag{18}$$

from the Lagrange's multipliers method we have

$$\frac{\partial U}{\partial V_i} + \beta \frac{\partial Z}{\partial V_i} = 0 \;\; ; \;\; i = 1, \ldots, N \tag{19}$$

where β is the multiplier. Writing the internal energy as

$$U = \frac{1}{2}(\mathbf{V}, \mathbf{C}\mathbf{V}) = \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N} V_k C_{kj} V_j \tag{20}$$

replacing Eqs. (18, 20) into Eq. (19) and using the symmetry of the matrix, we find

$$\sum_{j=1}^{N} C_{ij} V_j = \beta C_{i,N+1} \;\; ; \;\; i = 1, \ldots, N \tag{21}$$

and applying a sum over i on Eq. (21)

$$\sum_{j=1}^{N} V_j \sum_{i=1}^{N} C_{ij} = \beta \sum_{i=1}^{N} C_{i,N+1} \, , \tag{22}$$

$$-\sum_{j=1}^{N} V_j C_{N+1,j} = -\beta C_{N+1,N+1} \tag{23}$$

where we have used (5). Substracing Eqs. (23, 17) and solving for β we find

$$\beta = -\frac{Q_0}{C_{N+1,N+1}} \tag{24}$$

Eq. (21) can be rewritten as

$$\mathbf{C}\mathbf{V} = \beta \mathbf{v}_c \;\; ; \;\; \mathbf{v}_c^T \equiv (C_{1,N+1}, C_{2,N+1}, ..., C_{N,N+1}) \tag{25}$$

For a given β, the solution of Eq. (25) is unique because the r-matrix $\mathbf{C}$ is invertible. It is easy to check that $\mathbf{V} = (V_a, ..., V_a)$, is a solution of Eq. (25), inserting this solution in Eq. (25), we get

$$\begin{aligned} V_a \sum_{j=1}^{N} C_{ij} &= \beta C_{i,N+1} \, , \; i = 1, \ldots, N \\ -V_a C_{i,N+1} &= \beta C_{i,N+1} \, , \; i = 1, \ldots, N \end{aligned} \tag{26}$$

where we have used (5). From (26) we have

$$V_a = -\beta \tag{27}$$

Thus, the configuration of N voltages that minimizes the energy with a fixed value of Q_{int}, is given by

$$\mathbf{V}^T = (-\beta, -\beta, ..., -\beta) \tag{28}$$

This kind of solution for $\mathbf{V}$ is expected because the configuration of minimal energy is obtained when we connect all the internal conductors among them by conducting wires, this procedure clearly keeps Q_{int} constant and equates internal potentials. Since all potentials of the interior conductors are the same, we

[¶]This is in turn related with the fact that the null eigenvalue of the e-matrix is non-degenerate. If a degeneration of the null eigenvalue were present, we would have at least one eigenvector associated with the zero eigenvalue and linearly independent of ϕ_0. The existence of this eigenvector would imply the existence of a configuration of different potentials with a null value of the internal energy.

[‖]Since $\mathbf{u}^{(k)}$ are dimensionless, V_0 has units of potential. Note that when $\mathbf{V}$ is parallel to a principal axis (i.e. becomes an eigenvector of $\mathbf{c}$), all observables become simpler as in the case of the axis of rotation in the rigid body motion.

can define a single voltage between the external conductor and the internal ones, this voltage is $|\beta|$. Since $Q_{int} = -Q_{N+1} = Q_0$ we can figure out the system as equivalent to a system consisting of two conductors with charges $\pm Q_0$ and voltage $|\beta|$. Thus, we are led naturally to an equivalent capacitance for this system of potentials and charges

$$|Q_0| = C_{eq}\,|\beta| \Rightarrow C_{eq} = \left|\frac{Q_0}{\beta}\right| = C_{N+1,N+1}$$

where we have used Eq. (24). It can be checked that the internal energy U for the configuration described by (28) is

$$U = \frac{1}{2}C_{N+1,N+1}V_a^2 = \frac{1}{2}C_{eq}\beta^2 \tag{29}$$

as expected. A brief comment with respect to the e-matrix is in order. This matrix has no additional degrees of freedom with respect to the r-matrix so that we can formally write all results in terms of the elements C_{ij} of the r-matrix. Notwithstanding, the extended elements could be useful for explicit calculations. Assume for instance that for the problem in the present section, we want to calculate the total internal charge for a given voltage of the system, and the equivalent capacitance. These calculations can be done with the following expressions

$$\begin{aligned} Q_{int} &= \sum_{i=1}^{N} Q_k = V_a \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij} = C_{eq}V_a \\ C_{eq} &= C_{N+1,N+1} = -\sum_{i=1}^{N} C_{i,N+1} = \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij} \end{aligned}$$

Therefore, in terms of the elements of the r-matrix all the $N(N+1)/2$ coefficients must be evaluated with Eq. (3), to calculate Q_{int} and C_{eq}. In contrast, by using the e-matrix, only the $C_{N+1,N+1}$ coefficient should be calculated through Eq. (3) to find such observables. This difference becomes more important as N increases. Similar advantages appear in more general contexts (see appendix A of Ref. [27]).

7 The case of a chain of embedded conductors

Section 3 shows that the e-matrix is singular positive and the r-matrix is positive-definite. The first fact was independent of the connectivity of V_{S_T}. In contrast, the second fact was derived from the statement that all non-diagonal C_{ij} elements were strictly negative. However, it is shown in appendix C that in the case of a chain of embedded conductors (see Fig. 2), some of the non-diagonal elements C_{ij} are null because the volume V_{S_T} is disconnected. Then we should check whether the r-matrix is still positive-definite for the chain of embedded conductors.

Appendix C shows that elements of the form $C_{i,i\pm1}$ are non-zero in general. Appealing to an argument analogous to the one presented in Sec. 3 we can show that for a well-behaved geometry of our embedded conductors, $C_{ii} > 0$ and $C_{i,i\pm1} < 0$, while the remaining elements vanish*. With these properties and the fact that the sum of elements of each row of the e-matrix is null, we see that the sum of elements in each row of the r-matrix is null except for the $N-$row, for which the sum is positive. Therefore, the r-matrix of a chain of embedded conductors satisfies the conditions of theorem **C** in appendix A. Consequently, for a chain of embedded conductors, the r-matrix continues being positive-definite, and the e-matrix is still singular positive. Combining these facts with theorem **B** of appendix A, we obtain that the null eigenvalue of the e-matrix is non-degenerate†. It is again consistent with the fact that the only configuration of null internal energy is the one associated with all conductors at the same potential.

8 Conclusions

We have studied an electrostatic system consisting of a set of N conductors with an equipotential surface that encloses them. The associated matrix of capacitance has dimensions $(N+1) \times (N+1)$ (extended or e-matrix) even if the equipotential surface goes to infinity. It is usual in the literature to work with the matrix of dimension $N \times N$ (restricted or r-matrix), this practice is correct only if the voltage of the conductors is taken with respect to the potential of the equipotential surface. We prove that the e-matrix is a real positive and singular matrix, this is consistent with the fact that gauge invariance requires the existence of a null non-degenerate eigenvalue of this matrix. The r-matrix is a real positive-definite matrix so all its eigenvalues are positive.

By constructing a "potential space" with inner product, we can derive some results such as the reciprocity theorem and the non negativity of the electrostatic internal energy of the system from another point of view. The eigenvectors of the r-matrix correspond to the sets of voltages for which such voltages and their associated charges on each internal conductor are related with the same constant of proportionality, the positivity of the eigenvalues guarantees that those charges and voltages are of the same sign. In addition, a given eigenvalue is proportional to the internal energy associated with the set of voltages generated by its corresponding eigenvector. Moreover, a complete set of orthonormal eigenvectors of the r-matrix defines principal axes in the "potential space".

The problem of the minimization of the internal energy is studied under the constraint of constant value of the total internal charge. In this case we can define an equivalent capacitance for any number of internal conductors. From this problem, we realized that although the e-matrix has the same degrees of freedom as the r-matrix, such extension could lead to great simplifications of some practical calculations.

Further, systems of successive embbeded conductors are analyzed showing that some coefficients of capacitance are null for this system, allowing an important simplification for practical calculations. This fact is related with connexity properties of the volume in which Laplace's equation is considered. Moreover, we prove that for these configurations of embedded conductors the e-matrix is still positive singular with a non-degenerate eigenvalue and the r-matrix is positive-definite.

Finally, the properties of the matrix of capacitance shown here, can be useful for either a formal understanding or practical calculations in electromagnetism. It worths observing the similarity in structure between the matrix of capacitance and the inertia tensor.

*Of course if $i = 1$ (or $i = N+1$) the element $C_{i,i-1}$ (or $C_{i,i+1}$) does not exist. For a given i, at least one of them exists.

†Note that in this case, theorem **A** of appendix A cannot be used to establish the non-degeneration of the null eigenvalue.

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A Some special types of matrices

This appendix concerns the study of a special type of matrices. Let define Σ_k as the sum of the elements on the $k-$row of a given matrix. We shall make the following

Definition: A sp-matrix, is a square real matrix of finite dimension in which $c_{mk} = c_{km} \leq 0$ for $k \neq m$, and in which Σ_k is non-negative for all k. We denote with a single prime $\{k'\}$ the set of indices for which $\Sigma_{k'} > 0$ and with double prime $\{k''\}$ the set of indices for which $\Sigma_{k''} = 0$. If no prime is used, either situation could happen.

Theorem A: If $\mathbf{C}$ is a sp-matrix, then $\mathbf{C}$ is a positive matrix with respect to the usual complex inner product. ❶ If $\{k'\}$ is empty, the matrix is singular and vectors of the form $\mathbf{N}_0^T = (n, \ldots, n)$ are eigenvectors of $\mathbf{C}$ with null eigenvalue. Further, the null eigenvalue is non-degenerate if all elements of the matrix are non-null. ❷ If $\{k''\}$ is empty, the matrix is positive-definite.

Proof: We should prove that

$$\mathbf{V}^{\dagger}\mathbf{C}\mathbf{V} \geq 0 \tag{30}$$

for an arbitrary vector $\mathbf{V}$, and we should look under what conditions exists at least one non-zero vector $\mathbf{V}$ for which this bilinear expression is null. Rewriting $\mathbf{V} = \mathbf{N} + i\mathbf{M}$ with $\mathbf{N}$, $\mathbf{M}$ being real vector arrangements, and using the symmetry of $\mathbf{C}$, the bilinear form in Eq. (30) becomes $\mathbf{N}^T\mathbf{C}\mathbf{N} + \mathbf{M}^T\mathbf{C}\mathbf{M}$. Therefore, it suffices to prove the positivity (or non negativity) of the bilinear form with real vector arrangements. Let $\mathbf{N}$ be a non-zero real vector, the associated bilinear form is

$$\mathbf{N}^T\mathbf{C}\mathbf{N} = \sum_k n_k c_{kk} n_k + \sum_k \sum_{m \neq k} c_{km}\; n_k n_m$$

for the remaining of this appendix, we assume **that indices labeled with different symbols are strictly different**. We rewrite the bilinear form as

$$\begin{aligned}
\mathbf{N}^T\mathbf{C}\mathbf{N} &= \sum_k \{c_{kk} n_k^2 \\
&\quad + \frac{1}{2}\sum_m c_{km}[n_k^2 + n_m^2 - (n_k - n_m)^2]\} \\
\mathbf{N}^T\mathbf{C}\mathbf{N} &= \sum_k \left\{\left[c_{kk} + \frac{1}{2}\sum_m c_{km}\right] n_k^2 + \frac{1}{2}\sum_m c_{km} n_m^2\right\} \\
&\quad - \frac{1}{2}\sum_k \sum_m c_{km}\,(n_k - n_m)^2
\end{aligned} \tag{31}$$

Now, since $\Sigma_k = c_{kk} + \sum_m c_{km} \geq 0$, we have

$$c_{kk} + \frac{1}{2}\sum_m c_{km} \geq -\frac{1}{2}\sum_m c_{km} \tag{32}$$

it is convenient to separate the sets $\{k'\}$ and $\{k''\}$ in Eq. (32)

$$c_{k''k''} + \frac{1}{2}\sum_m c_{k''m} = -\frac{1}{2}\sum_m c_{k''m} \tag{33}$$

$$c_{k'k'} + \frac{1}{2}\sum_m c_{k'm} > -\frac{1}{2}\sum_m c_{k'm} \tag{34}$$

we examine first the case in which $\{k'\}$ is empty. In that case there are no equations of the type (34), and all indices accomplish the equation (33). Using (33) in Eq. (31) and the fact that $c_{km} \leq 0$, we find

$$\begin{aligned}
\mathbf{N}^T\mathbf{C}\mathbf{N} &= \sum_k \left\{\left[-\frac{1}{2}\sum_m c_{km}\right] n_k^2 + \frac{1}{2}\sum_m c_{km} n_m^2\right\} \\
&\quad + \frac{1}{2}\sum_k \sum_m |c_{km}|\,(n_k - n_m)^2
\end{aligned}$$

using the symmetry of the matrix and taking into account that k, m are dumb indices, the first two terms on the right-hand side vanish and we find

$$\mathbf{N}^T\mathbf{C}\mathbf{N} = \frac{1}{2}\sum_k \sum_m |c_{km}|\,(n_k - n_m)^2 \geq 0 \tag{35}$$

Equation (35) shows that the bilinear form is always non-negative and that $\mathbf{N}^T\mathbf{C}\mathbf{N} = 0$ for non-zero vector arrangements of the form $\mathbf{N}_0^T \equiv (n, n, \ldots, n)$. Consequently, the matrix is singular positive. We can check that $\mathbf{N}_0$ is an eigenvector of $\mathbf{C}$, with null eigenvalue. If all elements c_{km} are non-null, Eq. (35) shows that this is the only linearly independent solution, so that the zero eigenvalue is non-degenerate.

Now we examine the case in which $\{k''\}$ is empty, so there are no equations of the type (33), and all indices accomplish the equation (34). Replacing (34) into Eq. (31), using the symmetry of $\mathbf{C}$, the fact that $c_{km} \leq 0$, and that $\mathbf{N} \neq 0$ we find

$$\mathbf{N}^T\mathbf{C}\mathbf{N} > \frac{1}{2}\sum_k \sum_m |c_{km}|\,(n_k - n_m)^2 \geq 0 \tag{36}$$

and the bilinear form becomes positive if and only if $\mathbf{N} \neq 0$. Hence, the sp-matrix is positive definite when $\{k''\}$ is empty. **QED**.

Theorem B: Let $\mathbf{C}$ be a matrix of dimension $(N+1) \times (N+1)$, such that $\Sigma_i = 0$ for all rows. This matrix has eigenvectors of the form $\mathbf{N}_0^T = (n, \ldots, n)$ associated with a null eigenvalue. Let $\mathbf{C}_r$ be the $N \times N$ submatrix of $\mathbf{C}$ consisting of the elements C_{ij} of $\mathbf{C}$ with $i, j = 1, \ldots, N$. If $\mathbf{C}_r$ has no null eigenvalues[‡], the null eigenvalue of $\mathbf{C}$ is non-degenerate.

Proof: The condition $\Sigma_i = 0$ for $i = 1, \ldots, N+1$ gives

$$\sum_{k=1}^{N+1} C_{ik} = 0 \quad ; \quad i = 1, \ldots, N+1 \tag{37}$$

Eigenvectors of $\mathbf{C}$ with null eigenvalues must give

$$\sum_{k=1}^{N+1} C_{ik} n_k = 0 \quad ; \quad i = 1, ..., N+1 \tag{38}$$

assuming $n_k = n$ for all k and using condition (37), Eq. (38) is satisfied. Thus, $\mathbf{N}_0^T$ is an eigenvector associated with a null eigenvalue. From the condition (37) we also find

$$\begin{aligned}
\sum_{k=1}^{N+1} C_{ik} n_k &= \sum_{k=1}^{N} C_{ik} n_k + C_{i,N+1} n_{N+1} \\
&= \sum_{k=1}^{N} C_{ik} n_k + \left(-\sum_{k=1}^{N} C_{ik}\right) n_{N+1} \\
\sum_{k=1}^{N+1} C_{ik} n_k &= \sum_{k=1}^{N} C_{ik}\,(n_k - n_{N+1})\,;\; i = 1, .., N+1
\end{aligned} \tag{39}$$

[‡]If $\mathbf{C}_r$ is a normal matrix (or if it can be brought to the canonical form), it is equivalent to say that $\mathbf{C}_r$ is non-singular.

replacing (39) in (38) the latter becomes

$$\sum_{k=1}^{N} C_{ik}\left(n_k - n_{N+1}\right) = 0 \ ; \quad i = 1, ..., N+1 \tag{40}$$

in particular Eq. (40) holds for $i = 1, ..., N$. With this restriction Eq. (40) becomes

$$\mathbf{C}_r \mathbf{V} = 0 \ ; \quad \mathbf{V} \equiv \left(n_1 - n_{N+1}, n_2 - n_{N+1}, ..., n_N - n_{N+1}\right) \tag{41}$$

since $\mathbf{C}_r$ has no null eigenvalues, the only solution for Eq. (41) is $\mathbf{V} = \mathbf{0}$. Hence the only type of solutions for $\mathbf{N}$ are of the form $\mathbf{N} = (n_{N+1}, n_{N+1}, ..., n_{N+1})$ which are all linearly dependent. Hence, the null eigenvalue is non-degenerate. It is immediate that these solutions satisfy Eq. (40) for $i = N+1$ as well. Note that $\mathbf{C}$ is not neccesarily symmetric or real. **QED.**

Theorem C: Let $\mathbf{C}$ be a $N \times N$ sp-matrix such that $\{k''\} = \{1, 2, \ldots, N-1\}$ and the terms

$$c_{i,i+1} = c_{i+1,i} \quad , \ c_{i,i-1} = c_{i-1,i} \quad ; \quad i = 2, ..., N-1$$

are non-zero, while the remaining non-diagonal terms vanish. Then $\mathbf{C}$ is positive-definite.

Proof: Assume $\mathbf{C}$ as singular and arrive to a contradiction. Replacing Eqs. (33, 34) in Eq. (31) we obtain

$$\begin{aligned} \mathbf{N}^T \mathbf{C} \mathbf{N} &\geq \sum_k \left\{ \left[-\frac{1}{2} \sum_m c_{km} \right] n_k^2 + \frac{1}{2} \sum_m c_{km} n_m^2 \right\} \\ &\quad - \frac{1}{2} \sum_k \sum_m c_{km} \left(n_k - n_m\right)^2 \end{aligned}$$

such a replacement also shows that the equality holds if $n_N = 0$, while the strict inequality holds if $n_N \neq 0$. Using the symmetry of the matrix and the fact that $c_{km} \leq 0$ we have

$$\mathbf{N}^T \mathbf{C} \mathbf{N} \geq \frac{1}{2} \sum_k \sum_m |c_{km}| \left(n_k - n_m\right)^2$$

Since $\mathbf{C}$ is singular, a non-trivial solution must exist for the bilinear form to be null. For this, the equality must hold in this relation, therefore $n_N = 0$. The term on the right written in terms of the non-zero elements of the matrix yields

$$\begin{aligned} \mathbf{N}^T \mathbf{C} \mathbf{N} &= \frac{|c_{12}|}{2} \left(n_1 - n_2\right)^2 + \sum_{k=2}^{N-1} \frac{|c_{k,k+1}|}{2} \left(n_k - n_{k+1}\right)^2 \\ &\quad + \sum_{k=2}^{N-1} \frac{|c_{k,k-1}|}{2} \left(n_k - n_{k-1}\right)^2 + \frac{|c_{N,N-1}|}{2} n_{N-1}^2 \end{aligned}$$

for this expression to be zero each term in these sums must be zero. Since all matrix elements involved in this expression are non-zero, the last sum says that $n_{N-1} = 0$, while the other sums say that $n_1 = n_2 = ... = n_{N-1}$. Since n_N was already zero, it shows that the only solution is the trivial one, contradicting the singularity of the matrix. **QED.**

B Some properties of the internal energy

The equation (16) for the internal energy u in "natural units" can be written in terms of voltages instead of potentials with the r-matrix. By using the fact that ϕ_0 is an eigenvector of $\mathbf{c}_e$ with null eigenvalue, and the hemiticity of $\mathbf{c}_e$, Eq. (16) becomes

$$\begin{aligned} 2u &= \left(\phi, \mathbf{c}_e \left(\phi - \phi_0\right)\right) = \left(\mathbf{c}_e \phi, \left(\phi - \phi_0\right)\right) \\ 2u &= \left(\mathbf{c}_e \left(\phi - \phi_0\right), \left(\phi - \phi_0\right)\right) = \left(\phi - \phi_0, \mathbf{c}_e \left(\phi - \phi_0\right)\right) \end{aligned}$$

defining a vector arrangement of $N+1$ voltages $\mathbf{V}_e$ we find

$$2u = \left(\mathbf{V}_e, \mathbf{c}_e \mathbf{V}_e\right) ; \quad \mathbf{V}_e^T \equiv \left(V_1, ..., V_N, V_{N+1}\right) , \quad V_i \equiv \varphi_i - \varphi_0$$

writing this bilinear form explicitly and expanding the sums we find

$$\begin{aligned} u &= \frac{1}{2} \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} c_{ij} V_i V_j \\ &= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} V_i V_j + V_{N+1} K_{N+1} \\ K_{N+1} &\equiv \sum_{i=1}^{N} c_{i,N+1} V_i + \frac{1}{2} c_{N+1,N+1} V_{N+1} \end{aligned}$$

choosing $\varphi_0 = \varphi_{N+1}$ we find $V_{N+1} = 0$, hence

$$\begin{aligned} u &= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} V_i V_j = \frac{1}{2} \left(\mathbf{V}, \mathbf{c} \mathbf{V}\right) \\ \mathbf{V}^T &\equiv \left(V_1, ..., V_N\right) \quad , \quad V_i \equiv \varphi_i - \varphi_{N+1} \end{aligned}$$

thus the internal energy can be written in terms of the r-matrix and the voltages in a simple way as long as the latter are defined with respect to the external potential φ_{N+1}. Since $\mathbf{V}$ is gauge invariant u also is.

On the other hand, remembering that we can always construct a complete orthonormal set of real dimensionless eigenvectors $\mathbf{u}_e^{(k)}$ of the e-matrix associated with the eigenvalues λ_k^e, we can write the internal energy associated with a configuration ϕ of potentials in terms of these eigenvalues and eigenvectors. Since the eigenvectors form a basis we can express ϕ as a linear combination of them

$$\phi = \sum_{m=1}^{N+1} b_m \mathbf{u}_e^{(m)} \quad ; \quad b_m \equiv \left(\mathbf{u}_e^{(m)}, \phi\right)$$

and Eq. (16) becomes

$$\begin{aligned} 2u &= \left(\phi, \mathbf{c}_e \phi\right) = \left(\sum_{m=1}^{N+1} b_m \mathbf{u}_e^{(m)}, \mathbf{c}_e \sum_{n=1}^{N+1} b_n \mathbf{u}_e^{(n)} \right) \\ &= \sum_{m=1}^{N+1} \sum_{n=1}^{N+1} b_m b_n \left(\mathbf{u}_e^{(m)}, \lambda_n^e \mathbf{u}_e^{(n)} \right) \\ &= \sum_{m=1}^{N+1} \sum_{n=1}^{N+1} \lambda_n^e b_m b_n \delta_{mn} = \sum_{n=1}^{N+1} \lambda_n^e b_n^2 \\ u &= \frac{1}{2} \sum_{n=1}^{N+1} \lambda_n^e \left| \left(\mathbf{u}^{(n)}, \phi \right) \right|^2 \end{aligned} \tag{42}$$

Since the eigenvectors $\mathbf{u}^{(k)}$ and the matrix $\mathbf{c}_e$ are dimensionless, the eigenvalues λ_k^e also are. It is straightforward to write this expression in terms of the r-matrix and the voltages with respect to φ_{N+1}

$$u = \frac{1}{2} \sum_{n=1}^{N} \lambda_n \left| \left(\mathbf{u}^{(n)}, \mathbf{V} \right) \right|^2 \tag{43}$$

where $\lambda_n, \mathbf{u}^{(n)}$ are eigenvalues and eigenvectors of the r-matrix

C Some properties of chains of embedded conductors

Let us study a set of $N+1$ conductors which are successivily embedded. We label them $k = 1, \ldots, N+1$ from the inner to

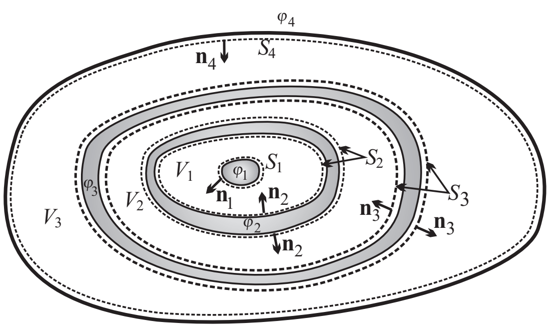

Figure 2: A chain of embedded conductors with $N = 3$. The surfaces S_2,S_3, have an inner and an outer part.

the outer. Observe that the surface S_k for each conductor with $k = 2, \ldots, N$ has an inner and an outer part, but for S_{N+1} we only define an inner part and for S_1 we only define an outer part (see Fig. 2). In addition, we define V_k with $k = 1, \ldots, N$ as the volume formed by the points exterior to the conductor k and interior to the cavity of the conductor $k + 1$ that contains the conductor k. Let us examine the non-diagonal elements C_{km} assuming from now on that $k < m$.

From Eq. (1) we see that if $m - k = 1$ then $f_m(S_k) = 0$ and $f_m(S_{k+1}) = 1$ because $S_{k+1} = S_m$, the volume V_k is precisely delimited by the outer part of the surface S_k and the inner part of the surface S_{k+1}; thus f_m has a non-trivial solution in V_k. Therefore, we have in general that $\nabla f_m \neq 0$ in V_k and in the surfaces that delimit it. Thus the integral

$$C_{km} = -\varepsilon_0 \oint_{S_k} \nabla f_m \cdot \mathbf{n}_k \, dS \tag{44}$$

has a contribution from the outer part of S_k. Now, if V_{k-1} exists (i.e. if $k > 1$), and taking into account that $f_m(S_{k-1}) = f_m(S_k) = 0$, the uniqueness theorem says that the only solution in V_{k-1} is $f_m = 0$ and hence $\nabla f_m = 0$ in this volume and in the surfaces that delimite such a volume[§]. Thus the integral surface in (44) has no contributions from the inner part of S_k.

Now, if $m - k \geq 2$ we see that $f_m(S_k) = f_m(S_{k+1}) = 0$, then the only solution in V_k is $f_m = \nabla f_m = 0$ in this volume and in the surfaces that delimite such a volume. Thus the integral surface in (44) has no contributions from the inner part of S_k. On the other hand, if V_{k-1} exists ($k > 1$), and since $f_m(S_{k-1}) = f_m(S_k) = 0$ we see once again that $f_m = \nabla f_m = 0$ in the volume V_{k-1} and in the surfaces that delimite it; so the integral (44) has no contribution from the outer part of S_k either.

From the previous discussion and appealing to the symmetry of the e-matrix, we conclude that $C_{km} = 0$ for $|m - k| \geq 2$. In addition, when $|m - k| = 1$, the surface integral (44) receives contribution only from the outer part of S_k. Notice that the previous behavior has to do with the fact that the total volume V_{S_T} consists of several disjoint (and so disconnected) regions and that $|k - m| \geq 2$ indicates that these labels are always associated with disjoint volumes. In the last discussion we have not included the possibility that the most interior conductor has a cavity. Since it would be an empty cavity, the surface and volume of this cavity do not contribute to the calculation of any coefficient of capacitance (see Ref. [27]).

From the results above, we see that for $N + 1$ successivily embedded conductors with $N \geq 2$, we have

$$\begin{aligned} C_{ii} &= -\left(C_{i,i-1} + C_{i,i+1}\right) \;\; ; \;\; i = 2, \ldots, N \\ C_{11} &= -C_{12} \;\; ; \;\; C_{N+1,N+1} = -C_{N,N+1} \end{aligned}$$

How many degrees of freedom do we have for the e-matrix?.

§Remember that the surfaces are slightly different from the surfaces of the conductors for the gradient to be well-defined.

D Suggested Problems

For checking the comprehension of the present formulation and its advantages, we give some general suggestions for the reader.

1. Show all the properties stated here for the r-matrix and e-matrix with specific examples.
2. Look for differences and similarities between the matrix of capacitance in electrostatics and the inertia tensor in mechanics, from the physical and mathematical point of view.
3. From $(\phi, \mathbf{c}\phi) \equiv k$ we find $(\mathbf{u}, \mathbf{cu}) = 1$ with $\mathbf{u} \equiv \phi/\sqrt{k}$. This defines the equation of an ellipsoid, describe how to find the length of the axes of the ellipsoid in the Φ^N and Φ^{N+1} spaces for the r-matrix and the e-matrix respectively. Describe the principal axes in these "potential spaces".
4. By setting $\partial u/\partial \varphi_i = 0$, prove that in the absence of constraints, the only local minimum of the internal energy is given by sets of the type ϕ_0.
5. Prove Eq. (29) for the minimal internal energy under the constraint of constant internal charge.
6. Let $\{a, b\}$ be two positive numbers. Consider the 4×4 matrix given by
$$\mathbf{C}_{4\times 4} = \begin{pmatrix} a\mathbf{B}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & b\mathbf{B}_{2\times 2} \end{pmatrix} \;\; ; \; \mathbf{B}_{2\times 2} \equiv \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
this is a sp-matrix in which the sum of elements in each row is zero. Further, $\lambda = 0$ is a two-fold degenerate eigenvalue of $\mathbf{C}$. Can $\mathbf{C}$ be a matrix of capacitance associated with a given electrostatic set of conductors?.
7. Look up for more applications of singular positive and positive-definite matrices in different contexts of Physics.

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